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Generalized Dependent Elements of Generalized Reverse Derivation on Semiprime Rings

Abstract- Let R be an associative ring, and $\check{S}:R \rightarrow R$ be a map, if there exists an element $e \in R$ such that $\check{S}(u)e = [u, e]e$, for every $u \in R$, in this case e is called Generalized Dependent Element of \check{S} , and $\check{G}-D(\check{S})$ denote the set of all Generalized Dependent Elements of \check{S} . In this paper the result proved, let R be semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation, related with derivation d , then $e \in \check{G}-D(F)$, iff, $e \in Z(R)$ and $eF(u) = 0$ for every $u \in R$.

Keywords- semiprime rings, derivation, generalized reverse derivation.

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1. Introduction

Through our paper, R is ring (associative) and $Z(R)$ denote center R . Recollection, a ring R is semiprime whenever $xRx = (0)$, then $x = 0$. An additive map $d: R \rightarrow R$ is named a derivation (resp. Jordan derivation) if $d(uv) = d(u)v + ud(v)$ (resp. $d(u^2) = d(u)u + ud(u)$) satisfy for every $u, v \in R$. Following [1] an additive map $d: R \rightarrow R$ named left derivation (resp. Jordan left derivation) if $d(uv) = ud(v) + vd(u)$ (resp. $d(u^2) = 2ud(u)$) for every $u, v \in R$. Obviously, every left derivation on a ring R is a Jordan left derivation, but the converse in general is not true. Ashraf and Ali in [2] defined the generalized left derivation as, an additive map $F: R \rightarrow R$ is named a generalized left derivation (resp. a generalized Jordan left derivation) related with Jordan left derivation, if there exists a Jordan left derivation $d: R \rightarrow R$, such that $F(uv) = uF(v) + vd(u)$ (resp. $F(u^2) = uF(u) + ud(u)$) for every $u, v \in R$. In [3] Vukman and Brešar defined the reverse derivation as, an additive map $d: R \rightarrow R$ satisfactory $d(uv) = d(v)u + vd(u)$, for every $u, v \in R$. It is clear, that reverse derivation and derivation are the same if R is commutative. In [4], [5] explored a more reverse derivations. The concept of generalized reverse derivation was introduced first in [6] as, an additive map $F: R \rightarrow R$ is named a generalized reverse derivation if we have a reverse derivation $d: R \rightarrow R$, satisfy: $F(uv) = F(v)u + vd(u)$, for every $u, v \in R$. Following Reddy and et al. in [7], work replacing of existence of reverse derivation by derivation in above definition of generalized reverse derivation in [6]. The research of dependent elements appeared in [8], by Thaheem and Laradji. Lately, several authors Ali, Chaudhry and others in [9, 10, 11, and 12] proved more results on dependent elements on

rings. Let us take $\check{S}:R \rightarrow R$ as a map, if there exists an element $e \in R$ such that $\check{S}(u)e = [u, e]e$, for every $u \in R$, in this case e is called Generalized Dependent Element of \check{S} , and $\check{G}-D(\check{S})$ denote the set of all Generalized Dependent Elements of \check{S} . In this paper the result proved, if R is semiprime ring, and F is a generalized reverse derivation related with derivation d , on R , then $e \in \check{G}-D(F)$, iff, $e \in Z(R)$ and $eF(u) = 0$ for every $u \in R$. In addition, we gave some results with semiprime ring of Generalized Dependent Elements for generalized reverse derivation related with derivation d .

2. The Results

Theorem.1

Suppose that R is semiprime ring, then

1- If there exist $x \in R$, such that $x[x, u] = 0$, for every $u \in R$, then $x \in Z(R)$, [13].

2- If there exist $x \in R$, such that $[x, u]x = 0$, for every $u \in R$, then $x \in Z(R)$, [6].

Theorem.2 [14]

Let R be a semiprime ring, and d be an inner derivation on R , if $[d(u), u] = 0$, for every $u \in R$, then $d = 0$.

Theorem.3

Suppose that R is a semiprime ring, and $F: R \rightarrow R$ is a generalized reverse derivation related with derivation d , then $d(u) \in Z(R)$, for every $u \in R$.

Proof: Let us take:

$$F(u^2v) = F(v)u^2 + vd(u^2), \text{ for every } u, v \in R \quad (1)$$

That is:

$$F(u^2v) = F(v)u^2 + vd(u)u + vud(u), \text{ for every } u, v \in R \quad (2)$$

Moreover, let us take:

$$F(u.uv) = F(uv)u + uvd(u), \text{ for every } u, v \in R \quad (3)$$

That get:

$$F(u^2v)=F(v) u^2+ vd(u)u + uvd(u), \text{ for every } u,v \in R \quad (4)$$

Comparing (2) and (4):

$$[u, v]d(u) = 0, \text{ for every } u, v \in R \quad (5)$$

Linearizing (6) on u:

$$[u, v]d(t)+ [t, v]d(u) = 0, \text{ for every } t, u, v \in R \quad (6)$$

Putting $v = vz$ in (5) and from (5), we obtain:

$$[u, v]zd(u) = 0, \text{ for every } t, u, v \in R \quad (7)$$

Also, Putting $z=d(t)z[t, v]$ in (7):

$$[u, v]d(t)z[t, v]d(u) = 0, \text{ for every } u, v, z, t \in R \quad (8)$$

From (6) and (8), we obtain:

$$[u, v]d(t)z[u, v]d(t) = 0, \text{ for every } t, u, v, z \in R \quad (9)$$

From hypothesis, R is semiprime, obtain:

$$[u, v]d(t) = 0, \text{ for every } u, v, t \in R \quad (10)$$

Theorem (2.1)(ii), give $d(t) \in Z(R)$, for every $t \in R$.

Theorem.4

Let R be semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation related with derivation d. Then $F(u) \in Z(R)$, for every $u \in R$.

Proof: our assumption, give:

$$F(uv^2)=F(v^2)u + v^2d(u), \text{ for every } v, u \in R \quad (1)$$

That is:

$$F(uv^2)= F(v)vu + vd(v)u + v^2d(u), \text{ for every } v, u \in R \quad (2)$$

Also,

$$F(uv^2)= F((uv)v)= F(v)uv + vd(uv), \text{ for every } v, u \in R \quad (3)$$

That is:

$$F(uv^2)= F(v)uv + vd(u)v + vud(v), \text{ for every } v, u \in R$$

(4) By Theorem 2.3, (4) give:

$$F(uv^2)= F(v)uv + v^2d(u) + vd(v)u, \text{ for every } v, u \in R \quad (5)$$

From (5), (2), we acquire:

$$F(v)vu= F(v)uv, \text{ for every } v, u \in R \quad (6)$$

That is:

$$F(v)[v, u] = 0, \text{ for every } v, u \in R \quad (7)$$

Putting $u=ru$ in (7), and using (7), leads to:

$$F(v)r[v, u] = 0, \text{ for every } r, v, u \in R \quad (8)$$

Furthermore, linearizing of relation (7), give:

$$F(v)[s, u] + F(s)[v, u] = 0, \text{ for every } s, u, v \in R \quad (9)$$

This implies:

$$F(v)[s, u] = -F(s)[v, u], \text{ for every } s, u, v \in R \quad (10)$$

Now, putting $r = [s, u]rF(s)$ in (8), and using (10), leads to:

$$F(s)[v, u]rF(s)[v, u] = 0, \text{ for every } s, u, v, r \in R \quad (11)$$

Because R is semiprime, (11) gives:

$$F(s)[v, u] = 0, \text{ for every } s, u, v \in R \quad (12)$$

Using Theorem (2.1)(i), we obtain $F(s) \in Z(R)$ for every s in R.

Corollary .5

Let R be semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation F related with

derivation d. Then F is a generalized left derivation with Jordan left derivation d.

Definition.6

Let $\check{S}:R \rightarrow R$ is a map, if there exists an element $e \in R$ such that $\check{S}(u)e = [u, e]e$, holds for every $u \in R$, in this case e is called Generalized Dependent Element of \check{S} , and $\check{G}-D(\check{S})$ denote the set of all Generalized Dependent Elements of \check{S} .

Examples .7

1)

Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in Z, \text{ the set of integer} \right\}$

Define map $\check{S}: R \rightarrow R$, as:

$$\check{S} \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

Take the element $e = \begin{bmatrix} a1 & 0 \\ 0 & 0 \end{bmatrix} \in R$

It is clear that, e is Generalized Dependent Element of \check{S} .

Note that if we take $e = \begin{bmatrix} a1 & b1 \\ 0 & c1 \end{bmatrix} \in R$, we show that, e is not Generalized Dependent Element of \check{S} .

2)

Let $R = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} : a, b \in Z, \text{ the set of integer} \right\}$

Define map $U: R \rightarrow R$, as:

$$U \left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Take the element $e = \begin{bmatrix} 0 & a1 & b1 \\ 0 & 0 & a1 \\ 0 & 0 & 0 \end{bmatrix} \in R$

It is clear that; e is Generalized Dependent Element of U.

Theorem.8

Let R be semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation F related with derivation d. Thus $e \in \check{G}-D(F)$, iff, $e \in Z(R)$ and $F(u)e=0$, for every $u \in R$

Proof: From hypothesis, $e \in \check{G}-D(F)$, thus:

$$F(u)e = [u, e]e, \text{ for every } u \in R \quad (1)$$

Putting $u = uv$ in (1), we have:

$$F(v)ue + vd(u)e = u [v, e] e + [u, e]ve, \text{ for every } u, v \in R \quad (2)$$

By Theorem 2.4, (2) gives:

$$uF(v)e + vd(u)e = u [v, e] e + [u, e]ve, \text{ for every } v, u \in R \quad (3)$$

Comparing (3) and (1), to obtain:

$$vd(u)e = [u, e]ve, \text{ for every } u, v \in R \quad (4)$$

Right multiplication of (4) by z, leads to:

$$vd(u)ez = [u, e]vez, \text{ for every } u, v, z \in R \quad (5)$$

Putting vz instead of v in (4), we get:

$$vzd(u)e = [u, e]vze, \text{ for every } z, u, v \in R \quad (6)$$

Use Theorem 2.3 in (6), and Subtracting (5) from (6), we get:

$$vd(u)[z, e] = [u, e]v[z, e], \text{ for every } z, u, v \in R \quad (7)$$

Left multiplication of (7) by u , leads to:

$$uvd(u)[z, e] = u[u, e]v[z, e], \text{ for every } z, u, v \in R \quad (8)$$

Putting $v = uv$ in (7), to obtain:

$$uvd(u)[z, e] = [u, e]uv[z, e], \text{ for every } z, u, v \in R \quad (9)$$

From (8) and (9), we get:

$$u[u, e]v[z, e] = [u, e]uv[z, e], \text{ for every } z, u, v \in R \quad (10)$$

This implies:

$$[u, [u, e]]v[z, e] = 0, \text{ for every } z, u, v \in R \quad (11)$$

Right multiplication of (11) by z , obtain:

$$[u, [u, e]]v[z, e]z = 0, \text{ for every } z, u, v \in R \quad (12)$$

Putting $v = vz$ in (11), we have:

$$[u, [u, e]]vz[z, e] = 0, \text{ for every } z, u, v \in R \quad (13)$$

Removing (12) from (13), we get:

$$[u, [u, e]]v[z, z, e] = 0, \text{ for every } z, u, v \in R \quad (14)$$

That get:

$$[u, [u, e]]v[u, [u, e]] = 0, \text{ for every } u, v \in R \quad (15)$$

By hypothesis, R is semiprime, implies:

$$[u, [u, e]] = 0, \text{ for every } u \in R \quad (16)$$

Defined the inner derivation $\Theta: R \rightarrow R$ as: $\Theta(u) = [u, e]$, is commuting, from Theorem 2.2, implies $[u, e] = 0$, for every $u \in R$, then $e \in Z(R)$

Further, from relation(1),we obtain $F(u)e = 0$ for every $u \in R$

$$\text{Conversely, } F(u)e = 0, \text{ for every } u \in R \quad (17)$$

Also, we have $e \in Z(R)$, this get:

$$[u, e] = 0, \text{ for every } u \in R \quad (18)$$

Right multiplication of (18) by e , we get:

$$[u, e]e = 0, \text{ for every } u \in R \quad (19)$$

From (17) and (19), one obtains:

$$F(u)e = 0 = [u, e]e, \text{ for every } u \in R$$

Corollary.9

Let R be semiprime ring, and $F: R \rightarrow R$ is a generalized reverse derivation F related with derivation d . Then $e \in \check{G}\text{-D}(F)$, if and only if, $e \in Z(R)$ and $eF(u) = 0$, for every $u \in R$.

Proof: From hypothesis $e \in \check{G}\text{-D}(F)$, thus by Theorem 2.8, $e \in Z(R)$ and $F(u)e = 0$

So, $e \in Z(R) \subseteq R$, that is $e \in R$, and by Theorem 2.4 ($F(R) \subseteq Z(R)$), we obtain $eF(u) = 0$, for every $u \in R$. (we can say $e \in Z(R)$, $F(R) \subseteq R$, we obtain $eF(u) = 0$, for every $u \in R$)

Conversely, let $eF(u) = 0$, by Theorem 2.4 ($F(R) \subseteq Z(R)$), implies

$$F(u)e = 0, \text{ for every } u \in R \quad (1)$$

And $e \in Z(R)$, that is:

$$[u, e] = 0, \text{ for every } u \in R \quad (2)$$

Right multiplication of relation (2) by e , we get:

$$[u, e]e = 0, \text{ for every } u \in R \quad (3)$$

From (1) and (3), one obtains:

$$F(u)e = 0 = [u, e]e, \text{ for every } u \in R \quad (4)$$

This implies $e \in \check{G}\text{-D}(F)$.

Corollary .10

Let R be semiprime ring, and $F: R \rightarrow R$ is a generalized reverse derivation F related with derivation d , if $e \in \check{G}\text{-D}(F)$, then $d(e) = 0$

Proof: From hypothesis, $e \in \check{G}\text{-D}(F)$, thus by Corollary 2.9, we obtain:

$$eF(u) = 0, \text{ for every } u \in R \quad (1)$$

Putting $u = ru$ in (1), one obtains:

$$eF(u)r + eud(r) = 0, \text{ for every } u, r \in R \quad (2)$$

From (1) and (2), one obtains:

$$eud(r) = 0, \text{ for every } u, r \in R \quad (3)$$

Putting $d(r)ue$ instead of u in (3), get:

$$ed(r)ue d(r) = 0, \text{ for every } u, r \in R \quad (4)$$

From hypothesis, get:

$$e d(r) = 0, \text{ for every } r \in R \quad (5)$$

Putting $r = d(r)$ in (5), get:

$$e d(d(r)) = 0, \text{ for every } r \in R \quad (6)$$

Also from(5),we get:

$$d(e d(r)) = 0, \text{ for every } r \in R \quad (7)$$

This gets:

$$d(e)d(r) + ed(d(r)) = 0, \text{ for every } r \in R \quad (8)$$

From (6) and (8), we get:

$$d(e)d(r) = 0, \text{ for every } r \in R \quad (9)$$

Putting re instead r in (9), get:

$$d(e)d(r)e + d(e)rd(e) = 0, \text{ for every } r \in R \quad (10)$$

From (9) and (10), we get:

$$d(e)rd(e) = 0, \text{ for every } r \in R \quad (11)$$

Again, R is semiprime, implies:

$$d(e) = 0 \quad (12)$$

Corollary.11

Let R be semiprime ring, and $F: R \rightarrow R$ is a generalized reverse derivation F related with derivation d , if $e \in \check{G}\text{-D}(F)$, then $F(e) = 0$

Proof: From hypothesis $e \in \check{G}\text{-D}(F)$, thus by Corollary 2.9, get:

$$eF(u) = 0, \text{ for every } u \in R \quad (1)$$

Also, Theorem 2.8, get:

$$F(u)e = 0, \text{ for every } u \in R \quad (2)$$

From (2), implies:

$$F(F(u)e) = 0, \text{ for every } u \in R \quad (3)$$

This implies:

$$F(e)F(u) + ed(F(u)) = 0, \text{ for every } u \in R \quad (4)$$

Also, From (1),

$$d(eF(u)) = 0, \text{ for every } u \in R \quad (5)$$

This implies:

$$d(e)F(u) + ed(F(u)) = 0, \text{ for every } u \in R \quad (6)$$

From Corollary 2.10:

$$ed(F(u)) = 0, \text{ for every } u \in R \quad (7)$$

From (4) and (7), get:

$$F(e)F(u)=0, \text{ for every } u \in R \tag{8}$$

Right multiplication of relation(8) by r , leads to:

$$F(e)F(u)r=0, \text{ for every } u, r \in R \tag{9}$$

Theorem 2.4, leads to:

$$F(e) r F(u)=0, \text{ for every } u, r \in R \tag{10}$$

For $u=e$ in (10) ,implies:

$$F(e) r F(e)=0, \text{ for every } r \in R \tag{11}$$

By hypothesis of R implies:

$$F(e)=0 \tag{12}$$

Corollary .12

Let R be semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation F related with derivation d , then $\check{G}\text{-D}(F)$ is semiprime commutative subring of R .

Proof: take $e \in \check{G}\text{-D}(F)$, thus Theorem 2.8, gives:

$$e \in Z(R) \tag{1}$$

This get:

$$eu=ue, \text{ for every } u \in R \tag{2}$$

Also, let $b \in \check{G}\text{-D}(F)$, by Theorem 2.8

$$b \in Z(R) \tag{3}$$

That is:

$$bu=ub, \text{ for every } u \in R \tag{4}$$

Subtracting (4) from (2), obtain:

$$(e-b)u=u(e-b), \text{ for every } u \in R \tag{5}$$

This get:

$$(e-b) \in Z(R) \tag{6}$$

Since the element e , and the element $b \in \check{G}\text{-D}(F)$, and from Theorem 2.8, we get:

$$F(u) e=0,$$

And

$$F(u) b=0, \text{ for every } u \in R \tag{7}$$

That is:

$$F(u) (e-b)=0, \text{ for every } u \in R \tag{8}$$

From (6), (8), and Theorem 2.8, we get:

$$(e-b) \in \check{G}\text{-D}(F) \tag{9}$$

Now since $b \in \check{G}\text{-D}(F)$, by Corollary 2.9, we get:

$$b F(u)=0, \text{ for every } u \in R \tag{10}$$

Multiply of (10) from the left by a , obtain:

$$eb F(u)=0, \text{ for every } u \in R \tag{11}$$

Right multiplication of (2) by b , we get:

$$eub=ueb, \text{ for every } u \in R \tag{12}$$

From (4), (12), we get:

$$ebu=ueb, \text{ for every } u \in R \tag{13}$$

That is:

$$eb \in Z(R) \tag{14}$$

By Corollary 2.9, and (11) and (14), gives:

$$eb \in \check{G}\text{-D}(F) \tag{15}$$

From (9) and (15), we get $\check{G}\text{-D}(F)$ subring of R .

Also, because $e, b \in Z(R) \subset R$, we get $eb=be$, that is $\check{G}\text{-D}(F)$ is commutative subring of R .

To prove $\check{G}\text{-D}(F)$ is semiprime, take $e \in \check{G}\text{-D}(F)$ $e=0$, $e \in \check{G}\text{-D}(F)$, Then:

$$eue = 0, \text{ for every } u \in \check{G}\text{-D}(F)$$

In particular, $e^3=0$

Thus we get $e=0$ (R is semiprime)

That is, $\check{G}\text{-D}(F)$ is semiprime.

Corollary.13

Let R be commutative semiprime ring, and $F:R \rightarrow R$ is a generalized reverse derivation F related with derivation d . then $\check{G}\text{-D}(F)$ is an ideal of R .

Proof: take $e, b \in \check{G}\text{-D}(F)$, and from (9) in Corollary 2.12 (see (1) to (9) in Corollary 2.12), we get:

$$(e-b) \in \check{G}\text{-D}(F) \tag{1}$$

Also, take $e \in \check{G}\text{-D}(F)$, and take $r \in R$, then:

$$F(u)e=0, \text{ for every } u \in R \tag{2}$$

Right multiply of (2) from the right by r , to get:

$$F(u)er=0, \text{ for every } u \in R \tag{3}$$

Also, Since $e \in Z(R)$, $r \in R$, then:

$$er=re \tag{4}$$

From (3),(4), obtain:

$$F(u)er= F(u)re=0, \text{ for every } u \in R \tag{5}$$

Now, Since R commutative:

$$er \in Z(R) \tag{6}$$

From (5), (6), and by Theorem 2.8, we get:

$$er=re \in \check{G}\text{-D}(F) \tag{7}$$

From (1), (7), we get $\check{G}\text{-D}(F)$ ideal

Remark .14:

Let R be semiprime ring, I is an ideal on R , then:

(i) I is semiprime subring of ring R

(ii)The center of I is contained in center of R .

References

[1] M. Ashraf and S. Ali, "On generalized Jordan left derivations in rings," Bull. Korean Math. Soc. 45 (2), 253-261, 2008.

[2] M. Ashraf, A. Ali and R. Rekha, "On generalized derivations of prime rings," South-East Bull. Math. 29 (4), 669-675, 2005.

[3] M. Brešar and J. Vukman, "On some additive mappings in rings with involution," A equations Math. 38, 178-185, 1989.

[4] M. Samman, "A note on reverse derivations," Inter. J. of Math. Edu. in Sci. and Tech. 37, 98-101, 2006.

[5] M. Samman and N. Al-Yamani, "Derivations and reverse derivations in semiprime rings," Inter. Math. Forum, 2, 1895-1902, 2007.

[6] S. B. Yass, "Strongly Derivation Pairs on Prime and Semiprime Rings," MSc. Thesis, Baghdad University, 2010.

[7] C.J. Reddy, G.V. Rao, S.V. Kuma "Generalized Reverse Derivations On Semi prime rings," Bull of Soc. Math. Serv. and Stand. 15, 1-4, 2015.

[8] A. Laradji and A.B. Thaheem "On dependent elements in semi prime rings", Math. Jap. 47, 576-584, 1998.

- [9] F. Ali and M. A. Chaudhry “Dependent elements of derivations, on semiprime rings,” *Inter. J. Math and Math. Sci* 10, 1-5, 2009.
- [10] J. Vukman, “On dependent elements and related problems in rings,” *Inter. Math. J.*, 6(2), 93–112, 2005.
- [11] C.J. Reedy, S.V. Kumar and G.V. Rao “Reverse Derivations On Semiprime rings,” *Inter. J. Eng. and Tech.* 3, 57-62, 2015.
- [12] N.N. Suliman and A.H. Majeed, “On dependent element and free action of derivation in Semi prime rings,” *Canad. J. pure and Applied Sci.*, 6(2), 2043-2047, 2012.
- [13] I.N. Herstein, “Rings with involution,” Univ. of Chicago Press, Chicago, 1976.
- [14] M. Brešar, “Centralizing mappings on von Neumann algebras,” *Proceedings of the Amer. Math. Soc.*, 111(2), 501–510, 1991.

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